Math 782: Riemannian Geometry

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About This Course

This course was taken in the Spring of 2022 at UNC Chapel Hill taught by Professor Yaiza Canzani. We used do Carmo. These notes were copied from the ones in my notebook and any mistakes are mine and not the lecturers.

Contents

1 Manifold Basics

Recall the tangent space $T_pM = \{\text{velocity vectors at p}\}$, where the velocity vectors act on functions $v[f] = \frac{d}{dt}(f \circ \gamma)(t)\big|_{t=0}$. (Insert picture)

We can build a basis of the tangent space as first starting with $p = (x_1, .., x_n) \in \mathbb{R}^n$, then $\frac{\partial}{\partial x_j}\Big|_p [f] = \frac{\partial f}{\partial x_j}(p)$, then we define $\frac{\partial}{\partial x_j}\Big|_p = x_{*,p}\left(\frac{\partial}{\partial x_j}\Big|_{x^{-1}(p)}\right)$. Notice here we're using the chart to get the left hand side to be in M and the rightside to be in $\mathbb{R}^n$.

For any curve $\gamma : (-\epsilon, \epsilon) \to M$ we can take $x^{-1} \circ \gamma$, which lives in $\mathbb{R}^n$ to get a way of defining a curve as a tangent vector (Insert picture):

$$\begin{aligned}\gamma'(0)[f] &= \frac{d}{dt}(f \circ \gamma)\big|_{t=0} \\ &= \frac{d}{dt}(f \circ x \circ (\gamma_1, ..., \gamma_n))\big|_{t=0} \\ &= \sum_j \frac{\partial f \circ x}{\partial x_j}(x^{-1}(\gamma(0)))\gamma_j'(0) \\ &= \sum_j \gamma_j'(0) \frac{\partial}{\partial x_j}\bigg|_{\gamma(0)}[f]\end{aligned}$$

The tangent bundle is the disjoint union of tangent spaces $TM = \{(p, v) : p \in M, v \in T_pM\} = \bigsqcup_p T_pM$. If we have an atlas $\{U_\alpha, x_\alpha\}$ for M, then we get an atlas for TM via $y_\alpha : U_\alpha \times \mathbb{R}^n \to TM$ via $y_\alpha(x, a_1, ..., a_n) = \left(x_\alpha(x), \sum a_i \frac{\partial}{\partial x_j^\alpha}\right)$

A vector field X on a manifold M associates to each point a tangent vector. So for any $p \in M$, $X(p) \in T_pM$. We take that $X : M \to TM$ is smooth, aka that X is a smooth section of the bundle $\pi X = id_M$. Throughout we take that $\Gamma(TM)$ to be the space of smooth sections (space of vector fields).

——————TO DO- FILL IN MORE BACKGROUND MATERIAL————————

2 Riemannian Manifolds

2.1 Basics

Recall that an inner product is a symmetric positive definite bilinear form.

Definition. A Riemannian Manifold is a pair (M, g) where M is a differentiable manifold, and g is a map that to each $p \in M$ associates a bilinear form $g(p) : T_pM \times T_pM \to \mathbb{R}$ that is non-degenerate, symmetric, positive definite.

We ask that for any $X, Y \in \Gamma(TM)$ the map $p \mapsto g(p)(X(p), Y(p))$ be smooth. This is akin to asking that g is a $(0, 2)$-tensor.

What does this look like in coordinates? Let $p \in M, (U, x)$ a coordinate chart that $p \in X(U)$, and let $(x_1, .., x_n) = x^{-1}(p)$. Then

$$g_{ij} = g(p)\left(\frac{\partial}{\partial x_i}\bigg|_p, \frac{\partial}{\partial x_j}\bigg|_p\right)$$

Then $g = (g_{ij})_{ij}$ as a matrix.

Example. In $\mathbb{R}^2$ if we take coordinates (x_1, x_2) then we have that

$$g_{\mathbb{R}^2}(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j} = \delta_{ij}$

Example. If we take polar coordinates $x(r,\theta) = (r\cos\theta, r\sin\theta)$. To find the metric we need to compute $\partial r, \partial\theta$. The outline of what the partial of θ looks like is written in the standard basis:

$$\left.\frac{\partial}{\partial\theta}\right|_p = --\left.\frac{\partial}{\partial x_1}\right|_p + --\left.\frac{\partial}{\partial x_2}\right|_p$$

Where we need to fill in the coefficients. To find what these are we need to evaluate the left side on a function:

$$\begin{aligned}\left.\frac{\partial}{\partial\theta}\right|_p[f] &= \frac{\partial}{\partial\theta}(f\circ x)\\ &= \frac{\partial x_1}{\partial\theta}\frac{\partial f}{\partial x_1} + \frac{\partial x_2}{\partial\theta}\frac{\partial f}{\partial x_2}\\ &= \left(-r\sin\theta\frac{\partial}{\partial x_1} + r\cos\theta\frac{\partial}{\partial x_2}\right)[f]\end{aligned}$$

And doing the same for ∂r we find

$$\left.\frac{\partial}{\partial r}\right|_p = \cos\theta\left.\frac{\partial}{\partial x_1}\right|_p + \sin\theta\left.\frac{\partial}{\partial x_2}\right|_p$$

So combining everything we arrive at

$$g_{\mathbb{R}^2}(r,\theta) = \begin{pmatrix}1 & 0\\ 0 & r^2\end{pmatrix}$$

Definition. If we have a map $\varphi : M \to N$ between manifolds, not necessarily the same dimension. φ is called a local isometry is

$$\varphi^* g_N(v,w) = g_N(\varphi_* v, \varphi_* w)$$

Theorem. Every manifold has a Riemannian metric

The proof is a consequence of either the Nash Embedding theorem or Whitney's theorem, basically you can fit any manifold into Euclidean space, even though the dimension may be massive.

Back to examples

Example. On S^2 using spherical coordinates

$$g_{S^2}(\theta,\phi) = \begin{pmatrix}\sin^2\phi & 0\\ 0 & 1\end{pmatrix}$$

A question one may ask is: Are there local immersions that yield local isometries? The answer is yes due to the Cartan-Janet theorem: If M^n is real analytic, then $k = n(n+1)/2$ the immersion is C^ω

Theorem (Nash Embedding)**.** M^n that is $C^m \implies M$ can be globally embedded in $\mathbb{R}^k$ yielding a global isometry if

- M is compact, $k = \frac{n(3n+11)}{2}$
- M is not compact $k = \frac{n(3n^2+7n+11)}{2} + 2n + 1$

2.2 Product Manifolds

A metric on $M \times N$ can be gotten by $g_{M\times N} := \pi_M^* g_M + \pi_N^* g_N$, where the π^* are pullbacks of projections

Example. $T^n = \prod^n S^1$. recall that $g_{S^1}(\theta) = d^2\theta$ with a coordinate chart $x(\theta) = (\cos\theta, \sin\theta)$, so $\sum_{i,j=1}^n g_{ij} d\theta_i \otimes d\theta_j$, hence $g_{S^1} = \sin^2\phi d\theta^2 + d\phi^2$.

Thus $g_{T^n} = \sum_j^n \pi^* g_{S^1}$, hence

$$\begin{aligned} g_{T^n}\left(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right) &= \pi_j^* g_{S^1}\left(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right) \\ &= g_{S^1}\left(\pi_j^*\frac{\partial}{\partial\theta_k}, \pi_j^*\frac{\partial}{\partial\theta_l}\right) \\ &= \delta_{jk}\delta_{jl} \end{aligned}$$

Therefore we see $g_{T^n} = d\theta_1^2 + \cdots + d\theta_n^2$

Note: γ' is a vector field along γ:

$$\gamma'(t_0)[f] = \frac{d}{dt}(f\circ\gamma)\Big|_{t=t_0} = \gamma_*\left(\frac{d}{dt}\Big|_{t=t_0}\right)[f]$$

Definition. $\gamma : [a,b] \to M$ then length$(\gamma) = \int_a^b ||\gamma'(t)||\, dt$

Definition. A manifold M is orientable if it admits an atlas $\{U_\alpha, x_\alpha\}_\alpha$ with the property that if $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$ then $\det(J(x_\beta^{-1} \circ x_\alpha)) > 0$

——INSERT PICTURE FROM NOTES——

Observe: $\{\frac{\partial}{\partial x_1^\alpha}, ..., \frac{\partial}{\partial x_n^\alpha}\}$ and $\{\frac{\partial}{\partial x_1^\beta}, ..., \frac{\partial}{\partial x_n^\beta}\}$ have the same orientation

If we have a metric $g_{ij} = (AA^T)_{ij}$ then $dV_g = \sqrt{\det g} dx_1 \wedge \cdots \wedge dx_n$.

Send $g \mapsto dV_g$ —-INSERT PICTURE—–

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM, then the dual basis is $e_1^* \wedge \cdots \wedge e_n^*$, and thus $dV_g(p) = e_1^* \wedge \cdots \wedge e_n^*$
Let (U_α, x_α) be a coordinate chart near p, then $e_1^* \wedge \cdots \wedge e_n^* = (?)dx_1^\alpha \wedge \cdots \wedge d_n^\alpha$ for some function ?. Send

$$x_1^\alpha, ..., x_n^\alpha \to \{\frac{\partial}{\partial x_1^\alpha}, ..., \frac{\partial}{\partial x_n^\alpha}\} \to \{dx_1^\alpha \wedge \cdots \wedge d_n^\alpha\}$$

Then, —-Insert equivalence of dual basis —–

Extend the volume form over the whole manifold:

$$\text{Vol}_g(A) = \int_A dV_g = \int_A \sum p_\alpha dV_g = \sum_a \int_{A\cap x_{\alpha(U_\alpha)}} p_\alpha \sqrt{\det g} dx_1^\alpha \wedge \cdots \wedge d_n^\alpha$$

Where

$$\begin{aligned}\sqrt{\det(g_{ij})^\alpha} &= e_1^* \wedge \cdots \wedge e_n^* \left(\frac{\partial}{\partial x_1^\alpha}, ..., \frac{\partial}{\partial x_n^\alpha} \right) \\ &= \det J_y e_1^* \wedge \cdots \wedge e_n^* \left(\frac{\partial}{\partial x_1^\alpha}, ..., \frac{\partial}{\partial x_n^\alpha} \right) \\ &= \det \left(\frac{\partial x_\beta^{-1} \circ x_\alpha}{\partial x} \right) \sqrt{\det(g_{ij})^\beta}\end{aligned}$$

The dV_g is well defined (preserved under change of coordinates)

Example. $\mathrm{Vol}(S^2) = \int_{S^2} dV_{g_{S^2}} = \int_0^{2\pi} \int_0^\pi \sin\phi \, d\phi d\theta = 4\pi$

3 Connections

—-INSERT PICTURE—–

$$\left.\frac{dY}{dt}\right|_t = \lim_{h \to a} \frac{Y(\gamma(t+h)) - Y(\gamma(t))}{h}$$

Note: This crucially uses $\mathbb{R}^n$: the subtraction

Obs:

$$\left.\frac{dY}{dt}\right|_t = \sum_{j=1}^N \frac{d}{dt} y_j(\gamma(t)) \left.\frac{\partial}{\partial u_j}\right|_{\gamma(t)}$$

Where $Y = \sum_{j=1}^N y_j \frac{\partial}{\partial u_j}$

Then $\frac{DY}{dt} = \pi\left(\frac{dY}{dt}\right)$ where $\pi : T_{\gamma(t)}\mathbb{R}^N \to T_{\gamma(t)}M$, and $T_{\gamma(t)}\mathbb{R}^N = T_{\gamma(t)}M \oplus \left(T_{\gamma(t)}M\right)^\perp$.

Consider $M \subset \mathbb{R}^N$, with coordinates $(u_1, ..., u_N)$ and $\gamma(t)$ be a curve on M. Define a vector field $Y(t) = \sum b_i(t) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}$. Let $(x_1, ..., x_n)$ be coordinates on M near the curve. Then

$$\frac{dY}{dt}(t) = \sum \dot{b}_i(t) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)} + b_i(t) \frac{d}{dt} \left(\left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)} \right)$$

Via the product rule. Now recall the definition of $\left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}$:

$$\left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)} = \sum^N \frac{\partial u_k}{\partial x_i}(x^{-1} \circ \gamma(t)) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}$$

Hence when we take a derivative we get

$$\frac{d}{dt} \left(\left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)} \right) = \sum_{k=1}^N \sum_{j=1}^n \gamma_j(t) \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} \left.\frac{\partial}{\partial u_k}\right|_{\gamma(t)}$$

So putting this together we get

$$\frac{dY}{dt}(t) = \sum \dot{b}_i(t)\frac{\partial}{\partial x_i}\bigg|_{\gamma(t)} + b_i(t)\sum_{k=1}^{N}\sum_{j=1}^{n}\gamma_j(t)\frac{\partial}{\partial x_j}\frac{\partial u_k}{\partial x_i}\frac{\partial}{\partial u_k}\bigg|_{\gamma(t)}$$

This $\frac{DY}{dt}$ is the projection of this to the tangent space of M at the point on the curve $\gamma(t)$ hence

$$\frac{DY}{dt} = \sum_{l=1}^{n}\dot{b}_l(t)\frac{\partial}{\partial x_l}\bigg|_{\gamma(t)} + \sum_{i,j} b_i(t)\gamma_j(t)'\Gamma_{ij}^l(\gamma(t))\frac{\partial}{\partial x_l}\bigg|_{\gamma(t)}$$

Where $\Gamma_{ij}^l = \sum_{k=1}^{N}\frac{\partial^2 u_k}{\partial x_i \partial x_j}A_k^l$, where $\pi\left(\frac{\partial}{\partial u_k}\Big|_{\gamma(t)}\right) = \sum_{l=1}^{n} A_k^l \frac{\partial}{\partial x_l}$

Definition. $\gamma : I \to M$, if $Y \in \Gamma(TM)$ then

$$\nabla_{\gamma'(t)}Y = \frac{DY}{dt}(t)$$

If $X \in \Gamma(TM)$ then $\nabla_X Y(p) = \nabla_{X(p)}Y = \nabla_{\gamma'(t)}Y$. In local coordinates:

$$\nabla_X Y = \sum_{l=1}^{n}\left(X(b_l) + \sum_{i,j} b_i a_j \Gamma_{ij}^l\right)\frac{\partial}{\partial x_l}\bigg|_{\gamma(t)}$$

Officially ∇ is a map: $\nabla : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$, where $\Gamma(TM)$ is sections on the tangent bundle

Properties:

- $\nabla_{fX+gY}Z = \nabla_{fX}Z + \nabla_{gY}Z = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_X(Z+Y) = \nabla_X Z + \nabla_X Y$
- $\nabla_X fY = X(f)Y + f\nabla_X Y$

Definition. A map $\nabla : \Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$ that satisfies the above properties is called an affine connection

Proposition. Let M be a manifold with an affine connection ∇, and let γ be a curve. Then there is a unique map that to each vector field V along γ associates a new vector field $\frac{DY}{dt}$ along the curve, such that

- $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$
- $\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt}$
- $\nabla_{\gamma'}Y = \frac{DY}{dt}$

Proof. Let $(x_1, .., x_n)$ be a coordinate chart in a neighborhood of $\gamma(t)$. Let $V(t) = \sum_{i=1}^{n} v_i(t)\frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$. By definition

$$\frac{DV}{dt}(t) = \sum_{i=1}^{n} v_i'(t)\frac{\partial}{\partial x_i}\bigg|_{\gamma(t)} + \sum v_i(t)\frac{D}{dt}\left(\frac{\partial}{\partial x_i}\bigg|_{\gamma(t)}\right)$$

Where $\frac{D}{dt}\left(\frac{\partial}{\partial x_i}\Big|_{\gamma(t)}\right) = \nabla_{\gamma'(t)}\frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$ so we can rewrite the above as

$$\frac{DV}{dt}(t) = \sum_{i=1}^{n} v_i'(t)\frac{\partial}{\partial x_i}\Big|_{\gamma(t)} + \sum v_i(t)\gamma_j'(t)\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}$$

This shows that $\frac{DV}{dt}$ is uniquely defined on $x_\alpha(U_\alpha)$, and is well defined on the intersection of patches as $\frac{DV}{dt}$ is well defined.

Remark: $\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} = \sum_{l=1}^{n}\Gamma_{ij}^{l}\frac{\partial}{\partial x_l}$, these Gamma's are called Christoffel symbols

If we take the above and express it in local coordinates we get

$$\frac{DV}{dt}(t) = \sum_{l=1}^{n}\left(v_l'(t) + \sum_{i,j}^{n} v_i(t)\gamma_j'(t)\Gamma_{ij}^{l}(\gamma(t))\right)\frac{\partial}{\partial x_l}\Big|_{\gamma(t)}$$

□

Example. In general if we have two vector fields $X = \sum_{i=1}^{n} a_i\frac{\partial}{\partial x_i}, Y = \sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i}$, then

$$\begin{aligned}\nabla_X Y &= \sum_{i=1}^{n} a_i\nabla_{\frac{\partial}{\partial x_i}}\left(\sum_{j=1}^{n} b_j\frac{\partial}{\partial x_j}\right)\\ &= \sum_{l=1}^{n}\left(\sum_{i=1}^{n} a_i\frac{\partial b_l}{\partial x_i} + \sum_{i,j=1}^{n} a_i b_k\Gamma_{ij}^{l}\right)\frac{\partial}{\partial x_l}\end{aligned}$$

Definition. A vector field V along a curve γ is said to be parallel if $\frac{DV}{dt}(t) = 0$

————-INSERT PICTURE——————

Proposition. Let $\gamma : I \to M$ be a differentiable curve, let $V_0 \in T_{\gamma(t)}M$. There exists a unique vector field V along γ such that $\frac{DV}{dt} = 0$ and $V(t_0) = V_0$. V is called the parallel transport of V_0 along γ

Proof. In coordinates we need to solve

$$\begin{cases} V_l' + \sum_{i,j=1}^{n} v_i\gamma_j'\Gamma_{ij}^{l} = 0 & \forall l = 1, ..., n\\ V_l(t_0) = V_0^{(l)} & \forall l = 1, ..., \end{cases}$$

The solution exists and is unique, it's a linear system of PDEs □

Definition. Let M be a manifold, ∇ an affine connection, g a Riemannian metric, we say ∇ is compatible with g is for every curve $\gamma : I \to M$, and any pair of parallel vector fields V, W along γ

$$\langle V, W\rangle_{g(\gamma(t))} \equiv constant$$

Proposition. (M, g) a Riemannian manifold, ∇ an affine connection. ∇ is compatible with g iff $\frac{d}{dt}\langle V, W\rangle = \langle\frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$ for any pair of vector fields V, W along γ

Proof. ($\leftarrow$) $\frac{d}{dt}\langle V, W\rangle = 0$ since $\frac{DV}{dt} = \frac{DW}{dt} = 0$, they're parallel vector fields
($\rightarrow$) Let $e_1(t_0), .., e_n(t_0)$ be an orthonormal basis of $T_{\gamma(t_0)}M$. Let $e_j(t)$ be the parallel transport of $e_j(t_0)$ along γ, $\langle e_j(t), e_i(t)\rangle = \langle e_j(t_0), e_i(t_0)\rangle$ by compatibility.
If $V = \sum_{i=1}^n v_i e_i, W = \sum_{j=1}^n w_j e_j$, then

$$\begin{aligned}\left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle &= \left\langle \sum_{i=1}^n v_i' e_i, \sum_{j=1}^n w_j e_j \right\rangle + \left\langle \sum_{i=1}^n v_i e_i, \sum_{j=1}^n w_j' e_j \right\rangle \\ &= \sum_{i=1}^n v_i' w_i + v_i w_i' \\ &= \frac{d}{dt} \sum v_i w_i \\ &= \frac{d}{dt} \langle V, W\rangle\end{aligned}$$

□

Proposition. ∇ is compatible with g iff $X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ for all vector fields

Definition. An affine connection ∇ is said to be symmetric if $\nabla_X Y - \nabla_Y X = [X, Y]$

Obs: In local coordinates

$$[X, Y] = \sum_{i,j=1}^n a_i \frac{\partial}{\partial x_i} b_j \frac{\partial}{\partial x_j} - \sum_{i,j=1}^n b_j \frac{\partial}{\partial x_j} a_i \frac{\partial}{\partial x_i}$$

Obs: Locally

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

Which means, in Christoffel symbols that $\Gamma_{ij}^l = \Gamma_{ji}^l$

Theorem (Fundamental Theorem of Riemannian Geometry)**.** Let (M, g) be a Riemannian manifold, then there exists a unique affine connection ∇ called the Levi-Civita Connection such that

- ∇ is compatible with the metric
- ∇ is symmetric

Proof. ——————————-FILL IN—————————— □

Obs: Christoffel Symbols: ————————-FILL IN————————-

$$\frac{DV}{dt} = \sum_{i=1}^n v_i' \frac{\partial}{\partial x_j} \quad \text{in } \mathbb{R}^n$$

4 Geodesics

——————Insert Picture—————— "Curves that look liks straight lines" aka $\frac{D\gamma'}{dt} = 0$

Obs: Geodesics have constant speed

$$\frac{d}{dt}\langle \gamma', \gamma'\rangle = \left\langle \frac{D\gamma'}{dt}, \gamma' \right\rangle + \left\langle \gamma', \frac{D\gamma'}{dt} \right\rangle = 0$$

Example. Take $\mathbb{H}$, then our metric is $g(x,y) = \frac{dx^2+dy^2}{y^2}$ the Christoffel symbols are $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$, and ....

The geodesic equations are $\gamma_k'' + \sum_{i,j=1}^m \Gamma^k_{ij}\gamma_i'\gamma_j' = 0$ ———-FILL IN———-

———INSERT PICTURE———— The geodesic flow on a manifold is $\frac{d}{dt}\phi^t(q) = X(\phi^t(q))$ ———–FILL IN THE MULTIPLE LECTURES—— ——————————————————

Corollary. If $\gamma : [a,b] \to M$ is parameterized by arc length, has the property that length$(\gamma) \leq$ length(β) where β is a piecewise differentiable curve joining $\gamma(a)$ to $\gamma(b)$ then γ is a geodesic

Proof. —FILL IN—- □

5 Curvature Tensors

The curvature R of a Riemannian manifold (M,g) is a map that to each pair $X, Y \in \Gamma(TM)$ associates another map $R(X,Y) : \Gamma(TM) \to \Gamma(TM)$ defined as

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

In local coordinates, e.g. in $\mathbb{R}^n$ we have that if $Z = \sum_{i=1}^n z_i \frac{\partial}{\partial x_i}$ then we know that $\nabla_X Z = \sum_{i=1}^n X(z_i)\frac{\partial}{\partial x_i}$ (recall the Christoffel symbols vanish in $\mathbb{R}^n$). Then we have

$$R(X,Y)Z = \sum_{i=1}^n YX(z_i)\frac{\partial}{\partial x_i} - \sum_{i=1}^n XY(z_i)\frac{\partial}{\partial x_i} + \sum_{i=1}^n [X,Y](z_i)\frac{\partial}{\partial x_i}$$

Thus in $\mathbb{R}^n$ $R(X,Y)Z = 0$ this is a 'flat space'
In general we have

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$$

so we get

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}$$

Notice that this somehow measure how much the covariant derivative commutes

Lemma. Let $f : A \subset \mathbb{R}^2 \to M$ be a parameterized surface, and let V be a vector field along f, then

$$R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})V = \frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V$$

Proof. Let $V = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, then $\frac{D}{\partial s} = \sum_i v_i \frac{D}{\partial s}\frac{\partial}{\partial x_i} + \sum_i \frac{\partial v_i}{\partial s}\frac{\partial}{\partial x_i}$. Thus

$$\frac{D}{\partial t}\frac{DV}{\partial s} = \sum_i v_i \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i} + \sum_i \frac{\partial v_i}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i} + \sum_i \frac{\partial v_i}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial x_i} + \sum_i \frac{\partial^2 v_i}{\partial t \partial s}\frac{\partial}{\partial x_i}$$

If we compute the covariant derivatives in reverse order we get

$$\frac{D}{\partial t}\frac{DV}{\partial s} - \frac{D}{\partial s}\frac{DV}{\partial t} = \sum_i v_i \left(\frac{D}{\partial t}\frac{DV}{\partial s} - \frac{D}{\partial s}\frac{DV}{\partial t}\right)\frac{\partial}{\partial x_i}$$

Now we have

$$\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i} = \sum_j \frac{\partial^2 f_i}{\partial t \partial s}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} + \sum_j \frac{\partial f}{\partial s}\nabla_{\sum_k \frac{\partial f_k}{\partial t}\frac{\partial}{\partial x_k}}\left(\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}\right)$$

——FINISH PROOF———— □

Proposition. Let $X, YZ \in \Gamma(TM), p \in M$ then if f is a parameterized surface such that

- $f(s_0, t_0) = p$
- $\frac{\partial f}{\partial s}(s_0, t_0) = X_p$
- $\frac{\partial f}{\partial t}(s_0, t_0) = Y_p$

Then

$$R(X_p, Y_p)Z_p = \lim_{s \to s_0 t \to t_0} \frac{T^{XY}_{s,t} Z_p - Z_p}{(s - s_0)(t - t_0)}$$

Where $T^{XY}_{s,t} Z_p = ------------------------$
INSERT PICTURE————————

Proof. content... □

Proposition.
- R is a bilinear map in $\Gamma(TM) \times \Gamma(TM)$
- $R(X, Y)$ is linear

Proposition. Bianchi Identity

Definition. $R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$

Proposition. SOME THINGS ABOUT THIS ——————

In local coordinates

$$\begin{aligned}
R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} &= \sum_{l=1}^{n} R^l_{ijk}\frac{\partial}{\partial x_l} \\
&= \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} \\
&= \nabla_{\frac{\partial}{\partial x_j}}\left(\sum_l \Gamma^l_{ik}\frac{\partial}{\partial x_l}\right) - \nabla_{\frac{\partial}{\partial x_i}}\left(\sum_l \Gamma^l_{jk}\frac{\partial}{\partial x_l}\right) \\
&= \sum_l \left(\frac{\partial}{\partial x_j}\Gamma^l_{ik}\frac{\partial}{\partial x_l} + \Gamma^l_{ik}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_i}\Gamma^l_{jk}\frac{\partial}{\partial x_l} + \Gamma^l_{jk}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_l}\right) \\
&= \sum_l \left(\frac{\partial}{\partial x_j}\Gamma^l_{ik} - \frac{\partial}{\partial x_i}\Gamma^l_{jk}\right)\frac{\partial}{\partial x_l} + \sum_{l,m}\left(\Gamma^l_{ik}\Gamma^m_{jl} - \Gamma^l_{jk}\Gamma^m_{il}\right)\frac{\partial}{\partial x_m} \\
&= \sum_m \left(\frac{\partial}{\partial x_j}\Gamma^m_{ik} - \frac{\partial}{\partial x_i}\Gamma^m_{jk} + \sum_l \left(\Gamma^l_{ik}\Gamma^m_{jl} - \Gamma^l_{jk}\Gamma^m_{il}\right)\right)\frac{\partial}{\partial x_m}
\end{aligned}$$

The term in parenthesis is the coefficient R^m_{ijk}

————————FILL IN PICTURE——————-

5.1 Sectional Curvature

Let $M \subset \mathbb{R}^3$ be a surface, then recall that Gaussian curvature is

$$K = \frac{\det II(p)}{\det I(p)} = \frac{\det S_{ij}}{\det g_{ij}}$$

Where S is the matrix of the shape operator. There is a connection between the Gaussian curvature and the curature tensor:

Proposition. $R(X, Y, X, Y) = K(p)$ for X, Y orthonormal in T_pM

Proof.

$$\begin{aligned} \nabla_Z^{\mathbb{R}^3} &= \nabla_Z^M X + \langle \nabla_Z^{\mathbb{R}^3} X, U \rangle U \\ &= \nabla_Z^M X - \langle X, \nabla_Z^{\mathbb{R}^3} U \rangle U \\ &= \nabla_Z^M X + \langle X, S(Z) \rangle U \end{aligned}$$

□

5.2 Ricci Curvature